

QUANTIZATION OF A COMPLEX HIGHER ORDER DERIVATIVE THEORY USING PATH INTEGRALS

CARLOS A. MARGALLI AND J. DAVID VERGARA

ABSTRACT. This work addresses the quantization of a self-interacting higher order time derivative theory using path integrals. To quantize this system and avoid the problems of energy not bounded from below and states of negative norm, we observe the following steps: 1) We extend the theory to the complex plane and in this sense we double the degrees of freedom. 2) We add a total derivative to fix the convenient boundary conditions. 3) We check that the complex structure is consistent. 4) To map from the complex space to the real space we introduce reality conditions as second class constraints and we check that the interactions do not generate more constraints. 5) We built the measure of the complex path integral and we show that including currents the theory is projected to a self-interacting real theory that is renormalizable.

1. INTRODUCTION

The aim of this paper is to study the quantization of interacting high order time derivative theories using path integrals [1, 2, 3, 4, 5, 6, 7, 8]. It is common knowledge that the quantization of these kind of systems cannot be done consistently if we add interactions, since the decouple ghosts reappear again in this instance [9]. Then, one is forced to select in these cases between a Hamiltonian no bounded from below, *i.e.* no vacuum state, or states with negative norm, *i.e.* ghosts [10, 11]. So it seems that there is no way to quantize consistently these systems. However, in recent years there are a renewal interest in these kind of theories, because they are an example of systems with Lorentz symmetry breaking [12, 13]. The usual way to treat these theories is to consider a perturbative approach [4, 6] or an effective approach [2, 3]. Both approaches have the disadvantage that in the procedure they lost degrees of freedom, *i.e.* at the end we only have a description of the low energy modes. If, we consider that these theories are only the effective part of a more fundamental theory, this kind of approach have a full justification. However, if we consider that these theories are fundamental we are losing physically relevant information of the theory. An alternative approach was proposed in [14] in order to avoid losing physically relevant information. However, this approach uses the gauge/gravity correspondence then until now has only been formulated in the case of Anti-de-Sitter space.

Date: January 10, 2014.

To capture this lost information in [15], was proposed a procedure to quantize *complex high order derivative theories* at the level of particles and in [16] this work was extended to field theory using a canonical quantization. In this paper we generalize this procedure using path integrals and we analyze the full consistency of the procedure. The key point of our method is not quantize directly the real high order time derivative theory. Instead of that we quantize a complexified version of the theory and we project this theory to the real space. To realize this projection we analyze the consistency of the problem in the context of Dirac's quantization of systems with constraints and we show what type of interactions can be introduced consistently in the model. The first step in the procedure is to complexify the original high order derivative theory. This implies that the original degrees of freedom of the theory are duplicated. The second step is to add a total derivative to the action, this derivative change the variables that are fixed at the boundary and consequence selects what fields are associated to the real degrees of freedom, this two steps are analyzed in section 2. In Section 3, we will look more closely at the complex structure of the theory and we check that the complexified equations of motion satisfy the Cauchy-Riemann conditions. The next step is to map the complex higher order derivative theory to a theory of real scalar fields. To make this mapping consistently we introduce a set of reality conditions that reduce the degrees of freedom of the complex theory and turn out the new theory real. We show that this reality conditions can be interpreted as second class constraints and we show that the free theory is closed under the evolution of this constraints. In Section 5 we introduce the interactions and we show that there are several types of interactions that are consistent with the reality conditions in the sense that the addition of these terms to the total Hamiltonian do not produce additional constraints, and in this sense we prove that our procedure is fully consistent inclusive when we add interactions. Section 6 is devoted to the quantization of the theory using path integrals. We introduce a measure in the extended space that includes the full set of second class constraints. This measure will project the complex theory to a real one. Because we add sources, to properly take into account the interactions. We add as additional conditions the relationships that we obtain for the sources from the equations of motion and reality conditions. To apply our results in Section 7 we quantize a high order derivative theory that corresponds to the Schwinger model via bosonization [17].

2. STRUCTURE OF A COMPLEX HIGHER ORDER DERIVATIVE THEORY

The Bernard-Duncan model is the most basic higher order time derivative field theory with action given by

$$(2.1) \quad S_0 = \int d^4x \frac{1}{2} [-(\Box\varphi)^2 + (m_1^2 + m_2^2)\partial_\mu\varphi\partial^\mu\varphi - m_1^2m_2^2\varphi^2]$$

where the scalar field φ is real. In this point one can think of quantizing this model following the usual procedure in quantum field theory, but we will find problems as the existence of negative norm states, the energy is unbounded from below and the dispersion matrix is non unitary. However, these troubles can be considered as interpretation failure following the ideas of Pais and Uhlenbeck [1] and the conclusions of Hawking and Hertog [5], taking into account to two independent Hilbert spaces. In spite of all, it is not possible to include interactions in the system.

In this work we want to take a step forward and analyze an extension to the complex plane of the Bernard-Duncan model. This extension implies that the theory is not Hermitian. However, we will show that this complex model can be consistently restricted to a real phase space. This restriction is implemented using second class constraints following the Dirac's formalism [18], and we show that the constraint surface is preserved by the inclusion of some kind of interactions.

Firstly we establish a complexification of the Bernard-Duncan theory, i.e., we define that the higher order field is complex

$$(2.2) \quad \phi \equiv \phi_R + i\phi_I.$$

With this complexification, the number of degrees of freedom is duplicated. Secondly we attach a total derivative term to the complex Lagrangian density, this does not modify the equations of motion, but it allows to pick out the boundary conditions in terms of the field ϕ and the acceleration $\ddot{\phi}$.

The Lagrangian density attaching the total derivative term is

$$(2.3) \quad S = \int d^4x \left\{ \frac{1}{2}[(\Box\phi)^2 + (m_1^2 + m_2^2)\partial_\mu\phi\partial^\mu\phi - m_1^2m_2^2\phi^2] + \partial_\mu\phi(\partial^\mu\Box\phi) \right\}$$

with a complex field ϕ and a Lagrangian density which is an complex analytic function.

The above (2.3) is used to obtain directly the respective momenta

$$(2.4) \quad \pi_0 = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\dot{\phi}} + \partial_\nu\partial_\sigma \frac{\partial\mathcal{L}}{\partial\partial_\nu\partial_\sigma\dot{\phi}}$$

$$(2.5) \quad \pi_1 = \frac{\partial\mathcal{L}}{\partial\ddot{\phi}} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\ddot{\phi}}$$

$$(2.6) \quad \pi_2 = \frac{\partial\mathcal{L}}{\partial\phi^{(3)}}$$

resulting explicitly

$$(2.7) \quad \pi_0 = \phi^{(3)} - 2\nabla^2\dot{\phi} + (m_1^2 + m_2^2)\dot{\phi}$$

$$(2.8) \quad \pi_1 = 0$$

$$(2.9) \quad \pi_2 = \dot{\phi}$$

which are separable in real and imaginary parts. From these momenta we obtain four constraints which later we will study using Dirac's theory of constraints [18, 19].

To solve the equations of motion of (2.3) we need to specify the initial conditions $\phi(\vec{x}, t = t_0)$, $\dot{\phi}(\vec{x}, t = t_0)$, $\ddot{\phi}(\vec{x}, t = t_0)$ and $\phi^{(3)}(\vec{x}, t = t_0)$. The above define the configuration space of the Lagrangian theory and it shows that we have 8 linearly independent solutions, because our theory is complex.

From the Ostrogradsky Hamiltonian description of the third order theory (2.3) [20], we define the independent fields

$$(2.10) \quad \phi = \phi, \quad \eta = \dot{\phi}, \quad \xi = \ddot{\phi},$$

that will be used to establish clearly the appearance of constraints.

Fields and momenta allow us to introduce the Hamiltonian theory that is complex, but it is not clear the way we introduce higher order quantities. In next subsection, by means of Schwinger action's principle we argue reasons to introduce these fields (2.10) and these momenta (2.4) that result in a consistent Hamiltonian theory.

2.1. The Schwinger Variational Principle. In order to show the effect of boundary conditions, we use the Schwinger variational principle [21], applied to the action (2.3) resulting

$$(2.11) \quad \delta S = \int d^4x -\frac{1}{2}\phi \{ \square(\square) + (m_1^2 + m_2^2)\square + m_1^2 m_2^2 \} \phi + \int d^3x \{ \pi_0 \delta\phi + \pi_2 \delta\xi \},$$

where we find that the fields ϕ and ξ are naturally fixed on the boundary and in this sense the definitions in (2.4) -(2.6) are consistent. Also, from the variation we see that there are only 2 complex degrees of freedom in the configuration space, 4 in the phase space $(\phi, \xi, \pi_0, \pi_2)$, instead of the 3 complex that we see in (2.10), and this implies that our theory has constraints.

The Hamiltonian density resulting from Ostrogradsky's Theory is

$$(2.12) \quad \mathcal{H} = \pi_1 \xi + \pi_0 \pi_2 + 2\pi_2 \nabla^2 \eta - (m_1^2 + m_2^2) \eta \pi_2 - \frac{1}{2} \xi^2 - \frac{1}{2} (\nabla^2 \phi)^2 + \xi \nabla^2 \phi \\ - \frac{(m_1^2 + m_2^2)}{2} \eta^2 + \frac{(m_1^2 + m_2^2)}{2} \nabla \phi \cdot \nabla \phi + \frac{m_1^2 m_2^2}{2} \phi^2 - \eta \nabla^2 \eta + (m_1^2 + m_2^2) \eta^2 \\ - \xi \nabla^2 \phi + (\nabla^2 \phi)^2 + \pi_0 \pi_2 - \eta \pi_0.$$

The Hamiltonian density (2.12) in terms of real and imaginary parts is

$$\begin{aligned}
 (2.13) \quad \mathcal{H}_R = & \pi_{1R}\xi_R + \pi_{0R}\pi_{2R} + 2\pi_{2R}\nabla^2\eta_R - (m_1^2 + m_2^2)\eta_R\pi_{2R} - \frac{1}{2}\xi_R^2 \\
 & - \frac{1}{2}(\nabla^2\phi_R)^2 + \xi_R\nabla^2\phi_R - \frac{(m_1^2 + m_2^2)}{2}\eta_R^2 + \frac{(m_1^2 + m_2^2)}{2}\nabla\phi_R \cdot \nabla\phi_R \\
 & + \frac{m_1^2 m_2^2}{2}\phi_R^2 - \eta_R\nabla^2\eta_R + (m_1^2 + m_2^2)\eta_R^2 - \xi_R\nabla^2\phi_R + (\nabla^2\phi_R)^2 \\
 & + \pi_{0R}\pi_{2R} - \eta_R\pi_{0R} \\
 & - [\pi_{1I}\xi_I + \pi_{0I}\pi_{2I} + 2\pi_{2I}\nabla^2\eta_I - (m_1^2 + m_2^2)\eta_I\pi_{2I} - \frac{1}{2}\xi_I^2 \\
 & - \frac{1}{2}(\nabla^2\phi_I)^2 + \xi_I\nabla^2\phi_I - \frac{(m_1^2 + m_2^2)}{2}\eta_I^2 + \frac{(m_1^2 + m_2^2)}{2}\nabla\phi_I \cdot \nabla\phi_I \\
 & + \frac{m_1^2 m_2^2}{2}\phi_I^2 - \eta_I\nabla^2\eta_I + (m_1^2 + m_2^2)\eta_I^2 - \xi_I\nabla^2\phi_I + (\nabla^2\phi_I)^2 \\
 & + \pi_{0I}\pi_{2I} - \eta_I\pi_{0I}], \\
 (2.14) \quad \mathcal{H}_I = & [\pi_{1R}\xi_I + \pi_{1I}\xi_R + \pi_{0R}\pi_{2I} + \pi_{0I}\pi_{2R} + 2\pi_{2R}\nabla^2\eta_I + 2\pi_{2I}\nabla^2\eta_R \\
 & - (m_1^2 + m_2^2)\eta_R\pi_{2I} - (m_1^2 + m_2^2)\eta_I\pi_{2R} - \xi_R\xi_I - \nabla^2\phi_R\nabla^2\phi_I \\
 & + \xi_R\nabla^2\phi_I + \xi_I\nabla^2\phi_R - (m_1^2 + m_2^2)\eta_R\eta_I + (m_1^2 + m_2^2)\nabla\phi_R \cdot \nabla\phi_I \\
 & + m_1^2 m_2^2 \phi_R \phi_I - \eta_R\nabla^2\eta_I - \eta_I\nabla^2\eta_R + 2(m_1^2 + m_2^2)\eta_R\eta_I \\
 & - \xi_R\nabla^2\phi_I - \xi_I\nabla^2\phi_R + 2(\nabla^2\phi_R)(\nabla^2\phi_I) \\
 & + \pi_{0R}\pi_{2I} + \pi_{0I}\pi_{2R} - \eta_R\pi_{0I} - \eta_I\pi_{0R}].
 \end{aligned}$$

In this part we have found a real phase space with 12 real degrees of freedom, but with the Schwinger variational method we have 8 degrees of freedom. In fact, it suggests which the Hamiltonian theory is incomplete and we need to incorporate the constraints and in the next section this problem is faced using the Dirac's Method. Furthermore, in the spirit of the Complex Hamiltonian in the next section we analyze how to introduce a complex symplectic structure in such way that the Hamiltonian equations satisfy identically the Cauchy-Riemann conditions and in this way the evolution of the theory be analytical.

3. HAMILTON'S EQUATIONS AND CAUCHY-RIEMANN EQUATIONS

In this section we study the complex Hamiltonian structure and we select the correct symplectic structure in such way the classical evolution respects the analyticity of the system.

3.1. Complex Structure. In this part of the work, we are going to determine the minimal element that produces the temporal evolution in this complex description. From the separation in real and complex parts (2.2), we can establish a deeper

analysis, in this case the Legendre transformation for the Hamiltonian density (2.12) is

$$(3.1) \quad \mathcal{L} = \dot{\phi}_R \pi_{0R} - \dot{\phi}_I \pi_{0I} + \dot{\eta}_R \pi_{1R} - \dot{\eta}_I \pi_{1I} + \dot{\xi}_R \pi_{2R} - \dot{\xi}_I \pi_{2I} - \mathcal{H}_R \\ + i[\dot{\phi}_I \pi_{0R} + \dot{\phi}_R \pi_{0I} + \dot{\eta}_I \pi_{1R} + \dot{\eta}_R \pi_{1I} + \dot{\xi}_R \pi_{2I} + \dot{\xi}_I \pi_{2R} - \mathcal{H}_I].$$

where we introduce two symplectic structures. One for the real part of the Hamiltonian density and another one for the imaginary part.

To begin with we define a new notation that establish a more compact description

$$(3.2) \quad \Theta_A^a = (\phi_R, \eta_R, \xi_R, \phi_I, \eta_I, \xi_I),$$

$$(3.3) \quad \Pi_{bB} = (\pi_{0R}, \pi_{1R}, \pi_{2R}, \pi_{0I}, \pi_{1I}, \pi_{2I}),$$

with the index a running from $a = (\phi, \eta, \xi)$ and $b = (\pi_0, \pi_1, \pi_2)$ and the subscripts $A, B = (R, I)$ run over the real and imaginary parts. From the variation of the Lagrangian density (3.1) we obtain

$$(3.4) \quad \dot{\Theta}_R^a = \frac{\partial \mathcal{H}_R}{\partial \Pi_{aR}} = \frac{\partial \mathcal{H}_I}{\partial \Pi_{aI}}, \quad \dot{\Theta}_I^a = \frac{\partial \mathcal{H}_I}{\partial \Pi_{aR}} = -\frac{\partial \mathcal{H}_R}{\partial \Pi_{aI}},$$

$$(3.5) \quad \dot{\Pi}_{aR} = \frac{\partial \mathcal{H}_R}{\partial \Theta_R^a} = \frac{\partial \mathcal{H}_I}{\partial \Theta_I^a}, \quad \dot{\Pi}_{aI} = -\frac{\partial \mathcal{H}_I}{\partial \Theta_R^a} = \frac{\partial \mathcal{H}_R}{\partial \Theta_I^a}.$$

This is the full set of Hamilton equations and this system satisfies the Cauchy-Riemann conditions and in consequence the evolution given by these equations is analytical. From the symplectic structure given in (3.1) the Poisson brackets are

$$(3.6) \quad \{\Theta_A^a(t, \vec{x}), \Pi_{bB}(t, \vec{x}')\} = \mathcal{J}_{AB} \delta_b^a \delta^3(\vec{x} - \vec{x}'),$$

where we have defined \mathcal{J}_{AB} as

$$\mathcal{J}_{AB} = \begin{cases} 1 & \text{si } A = B = R \\ 0 & \text{si } A \neq B \\ -1 & \text{si } A = B = I \end{cases}$$

From this expression we obtain the general definition

$$(3.7) \quad \{F, G\} = \int d^3x' \left(\frac{\delta F}{\delta \phi_R} \frac{\delta G}{\delta \pi_{0R}} - \frac{\delta F}{\delta \pi_{0R}} \frac{\delta G}{\delta \phi_R} \right) - \left(\frac{\delta F}{\delta \phi_I} \frac{\delta G}{\delta \pi_{0I}} - \frac{\delta F}{\delta \pi_{0I}} \frac{\delta G}{\delta \phi_I} \right) \\ + \left(\frac{\delta F}{\delta \eta_R} \frac{\delta G}{\delta \pi_{1R}} - \frac{\delta F}{\delta \pi_{1R}} \frac{\delta G}{\delta \eta_R} \right) - \left(\frac{\delta F}{\delta \eta_I} \frac{\delta G}{\delta \pi_{1I}} - \frac{\delta F}{\delta \pi_{1I}} \frac{\delta G}{\delta \eta_I} \right) \\ + \left(\frac{\delta F}{\delta \xi_R} \frac{\delta G}{\delta \pi_{2R}} - \frac{\delta F}{\delta \pi_{2R}} \frac{\delta G}{\delta \xi_R} \right) - \left(\frac{\delta F}{\delta \xi_I} \frac{\delta G}{\delta \pi_{2I}} - \frac{\delta F}{\delta \pi_{2I}} \frac{\delta G}{\delta \xi_I} \right).$$

It is important to mention that in the parenthesis(3.6) there are several terms with opposite signs, they appear as a natural consequence of the complex structure and Legendre transformation (3.1).

3.2. Constraints in the System. In order to analyze the dynamics of the system we will use the Poisson brackets of (3.7) and now we consider the Dirac's Theory of constraints in order to handle systematically the restrictions which we have found in the definition of the momenta.

From the momenta (2.7)-(2.9) we get four primary constraint since we have divide the real and imaginary parts

$$(3.8) \quad \begin{aligned} \gamma_1 &= \pi_{1R}, & \gamma_2 &= \pi_{1I}, \\ \gamma_3 &= \pi_{2R} - \eta_R, & \gamma_4 &= \pi_{2I} - \eta_I. \end{aligned}$$

These constraints satisfy

$$(3.9) \quad \{\gamma_a, \gamma_b\} = C_{ab} \delta(\vec{x} - \vec{x}')$$

and

$$(3.10) \quad C_{ab} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

with determinant given by

$$(3.11) \quad \det(C_{ab}) = 1.$$

The temporal evolution of constraints is accomplished through the Cauchy-Riemann equations and we find that by the analyticity, it is associated with either the real or imaginary parts of the complex Hamiltonian density

$$(3.12) \quad \dot{\gamma}_1 = \int d^3x' \{ \gamma_1, \mathcal{H}_R + \alpha_1 \gamma_1 + \alpha_2 \gamma_2 + \alpha_3 \gamma_3 + \alpha_4 \gamma_4 \}$$

$$= [-2\nabla^2 + (m_1^2 + m_2^2)] \gamma_3 + \pi_{0R} + \alpha_3 \approx 0,$$

$$(3.13) \quad \dot{\gamma}_2 = [-2\nabla^2 + (m_1^2 + m_2^2)] \gamma_4 + \pi_{0I} - \alpha_4 \approx 0,$$

$$(3.14) \quad \dot{\gamma}_3 = -\gamma_1 - \nabla^2 \phi_R - \alpha_1 \approx 0,$$

$$(3.15) \quad \dot{\gamma}_4 = -\gamma_2 - \nabla^2 \phi_I + \alpha_2 \approx 0,$$

where we see that these constraints form a complete set, since from (3.12)-(3.15) we obtain the Lagrange multipliers α 's and because the expression (3.9) this set is made of second class constraints. This of course implies that we pass from 12 to 8 real degrees of freedom of the complex phase space. This constraint (3.8) define a new symplectic structure through the Dirac's brackets

$$(3.16) \quad \begin{aligned} \{F(t, \vec{x}_0), G(t, \vec{x})\}^* &= \{F(t, \vec{x}_0), G(t, \vec{x})\} \\ &- \int dx' dx'' \{F(t, \vec{x}_0), \gamma_a(t, \vec{x}')\} C^{ab} \delta(\vec{x}' - \vec{x}'') \{\gamma_b(t, \vec{x}''), G(t, \vec{x})\}. \end{aligned}$$

From the definition (3.16) the new symplectic structure results

$$\begin{aligned}
(3.17) \quad & \{\phi_R(t, \vec{x}), \pi_{0R}(t, \vec{x}_0)\}^* = \delta^3(\vec{x} - \vec{x}_0) \\
& \{\phi_I(t, \vec{x}), \pi_{0I}(t, \vec{x}_0)\}^* = -\delta^3(\vec{x} - \vec{x}_0) \\
& \{\eta_R(t, \vec{x}), \xi_R(t, \vec{x}_0)\}^* = -\delta^3(\vec{x} - \vec{x}_0) \\
& \{\eta_I(t, \vec{x}), \xi_I(t, \vec{x}_0)\}^* = \delta^3(\vec{x} - \vec{x}_0) \\
& \{\xi_R(t, \vec{x}), \pi_{2R}(t, \vec{x}_0)\}^* = \delta^3(\vec{x} - \vec{x}_0) \\
& \{\xi_I(t, \vec{x}), \pi_{2I}(t, \vec{x}_0)\}^* = -\delta^3(\vec{x} - \vec{x}_0).
\end{aligned}$$

We will use this new symplectic structure for the following computations, and in this way we have incorporated the correct boundary conditions.

To quantize the system we need to promote the Dirac's brackets (3.17) to commutators and this procedure is quite simple since the matrix (3.10) is constant, then we can use the constraints (3.8) directly in the Hamiltonian and from them eliminate the variables $(\eta_R, \eta_I, \pi_{1R}, \pi_{1I})$.

In the reduced space the Hamiltonian density is given by

$$(3.18) \quad \mathcal{H}_C = \mathcal{H}_{CR} + i\mathcal{H}_{CI}$$

$$\begin{aligned}
(3.19) \quad \mathcal{H}_{CR} = & \pi_{0R}\pi_{2R} - \frac{1}{2}\xi_R^2 - \frac{(m_1^2 + m_2^2)}{2}\pi_{2R}^2 + \frac{m_1^2 m_2^2}{2}\phi_R^2 + \frac{1}{2}(\nabla^2 \phi_R)^2 \\
& + \frac{(m_1^2 + m_2^2)}{2}(\nabla \phi_R)^2 + \pi_{2R}\nabla^2 \pi_{2R} \\
& - [\pi_{0I}\pi_{2I} - \frac{1}{2}\xi_I^2 - \frac{(m_1^2 + m_2^2)}{2}\pi_{2I}^2 + \frac{m_1^2 m_2^2}{2}\phi_I^2 + \frac{1}{2}(\nabla^2 \phi_I)^2 \\
& + \frac{(m_1^2 + m_2^2)}{2}(\nabla \phi_I)^2 + \pi_{2I}\nabla^2 \pi_{2I}]
\end{aligned}$$

$$\begin{aligned}
(3.20) \quad \mathcal{H}_{CI} = & [\pi_{0R}\pi_{2I} + \pi_{0I}\pi_{2R} - \xi_R\xi_I - (m_1^2 + m_2^2)\pi_{2R}\pi_{2I} \\
& + m_1^2 m_2^2 \phi_R \phi_I + \nabla^2 \phi_R \nabla^2 \phi_I \\
& + (m_1^2 + m_2^2)\nabla \phi_R \cdot \nabla \phi_I + \pi_{2R}\nabla^2 \pi_{2I} + \pi_{2I}\nabla^2 \pi_{2R}].
\end{aligned}$$

In this way at this moment the dynamics of the system is given by the Hamiltonian density (3.18) with the symplectic structure (3.17).

The reduced Hamiltonian density written in terms of complex variables $(\phi, \xi, \pi_0, \pi_2)$ is

$$\begin{aligned}
(3.21) \quad \mathcal{H}_C = & \pi_0\pi_2 - \frac{1}{2}\xi^2 - \frac{(m_1^2 + m_2^2)}{2}\pi_2^2 + \frac{m_1^2 m_2^2}{2}\phi^2 + \frac{1}{2}(\nabla^2 \phi)^2 \\
& + \frac{(m_1^2 + m_2^2)}{2}(\nabla \phi)^2 + \pi_2\nabla^2 \pi_2.
\end{aligned}$$

The Hamiltonian density (3.21) is tightly related to the Bernard-Duncan Hamiltonian density [15, 2]. We can see it by means of the canonical transformation

$$(3.22) \quad \begin{aligned} \phi(\vec{x}, t) &= \phi(\vec{x}, t), & \pi_\phi(\vec{x}, t) &= \pi_0(\vec{x}, t) + \nabla^2 \pi_2(\vec{x}, t), \\ \pi_{\dot{\phi}} &= -\xi + \nabla^2 \phi(\vec{x}, t), & \dot{\phi}(\vec{x}, t) &= \pi_2(\vec{x}, t), \end{aligned}$$

that imply

$$(3.23) \quad \mathcal{H}_D = \pi_\phi \dot{\phi} - \frac{1}{2} \pi_\phi^2 - \frac{(m_1^2 + m_2^2)}{2} \dot{\phi}^2 + \frac{(m_1^2 + m_2^2)}{2} (\nabla \phi)^2 + \frac{m_1^2 m_2^2}{2} \phi^2 - \nabla \phi \cdot \nabla \pi_{\dot{\phi}},$$

where it should be taken into account that our theory is complex.

Now, it is possible to apply the four constraints (3.8) to the Lagrangian density (3.1) which confirm the Dirac brackets

$$(3.24) \quad \begin{aligned} \mathcal{L}_C &= \pi_0 \dot{\phi} + \pi_2 \dot{\xi} - \mathcal{H}_C = \pi_{0R} \eta_R - \pi_{0I} \eta_I + \pi_{2R} \dot{\xi}_R - \pi_{2I} \dot{\xi}_I - \mathcal{H}_{CR} \\ &\quad + i[\pi_{0R} \eta_I + \pi_{0I} \eta_R + \pi_{2R} \dot{\xi}_I + \pi_{2I} \dot{\xi}_R - \mathcal{H}_{CI}], \end{aligned}$$

and as we proceeded it is necessary to set a new compact notation

$$(3.25) \quad \Lambda_C^c = (\phi_R, \xi_R, \phi_I, \xi_I),$$

$$(3.26) \quad \Upsilon_{dD} = (\pi_{0R}, \pi_{2R}, \pi_{0I}, \pi_{2I}),$$

with $c = \phi, \xi$ for the superscript of Λ , $d = \pi_0, \pi_2$ for the subscript of Υ and $C, D = R, I$. From the variation of the Lagrangian density (3.24) result the Cauchy-Riemann and Hamilton equations

$$(3.27) \quad \dot{\Lambda}_R^a = \frac{\partial \mathcal{H}_{CR}}{\partial \Upsilon_{aR}} = \frac{\partial \mathcal{H}_{CI}}{\partial \Upsilon_{aI}}, \quad \dot{\Lambda}_I^a = \frac{\partial \mathcal{H}_{CI}}{\partial \Upsilon_{aR}} = -\frac{\partial \mathcal{H}_{CR}}{\partial \Upsilon_{aI}},$$

$$(3.28) \quad \dot{\Upsilon}_{aR} = \frac{\partial \mathcal{H}_{CR}}{\partial \Lambda_R^a} = \frac{\partial \mathcal{H}_{CI}}{\partial \Lambda_I^a}, \quad \dot{\Upsilon}_{aI} = -\frac{\partial \mathcal{H}_{CI}}{\partial \Lambda_R^a} = \frac{\partial \mathcal{H}_{CR}}{\partial \Lambda_I^a},$$

with the Dirac bracket

$$(3.29) \quad \begin{aligned} \{F, G\}^* &= \int d^3x' \left(\frac{\delta F}{\delta \phi_R} \frac{\delta G}{\delta \pi_{0R}} - \frac{\delta F}{\delta \pi_{0R}} \frac{\delta G}{\delta \phi_R} \right) - \left(\frac{\delta F}{\delta \phi_I} \frac{\delta G}{\delta \pi_{0I}} - \frac{\delta F}{\delta \pi_{0I}} \frac{\delta G}{\delta \phi_I} \right) \\ &\quad + \left(\frac{\delta F}{\delta \xi_R} \frac{\delta G}{\delta \pi_{2R}} - \frac{\delta F}{\delta \pi_{2R}} \frac{\delta G}{\delta \xi_R} \right) - \left(\frac{\delta F}{\delta \xi_I} \frac{\delta G}{\delta \pi_{2I}} - \frac{\delta F}{\delta \pi_{2I}} \frac{\delta G}{\delta \xi_I} \right) \end{aligned}$$

that follows directly from the Dirac bracket (3.16). Explicitly, the parentheses are

$$(3.30) \quad \{\Lambda_A^a(t, \vec{x}), \Upsilon_{bB}(t, \vec{x}')\}^* = \mathcal{I}_{AB} \delta_b^a \delta^3(\vec{x} - \vec{x}'),$$

where the matrix \mathcal{I}_{AB} is

$$\mathcal{I}_{AB} = \begin{cases} 1 & \text{si } A = B = R \\ 0 & \text{si } A \neq B \\ -1 & \text{si } A = B = I \end{cases}$$

4. REALITY CONDITIONS IN THE MODEL

At this moment our model is still complex and the idea now is to introduce conditions that project our system to the real space. The idea of reality conditions that lead to real theories was proposed initially by Ashtekar in the context of general relativity [22]. The reality conditions have been important in order to give a physical sense to a complex theory such that these conditions are used to cancel the effect of the imaginary part [15]. In the original formulation of Ashtekar these conditions were implemented through the scalar product. However, in the Ref. [23] it was shown that these conditions can be implemented as second class constraints. For our case this procedure is more useful, since in the quantization of the theory the reality conditions can be implemented directly in the path integral. Another important point is that the full set of reality conditions is fixed by the evolution of the system, that means that starting from a set of constraints we evolve this set to show if the dynamics is consistent with the reality conditions, if appear new constraints we include these new conditions and we finish when the algebra is closed under the evolution.

In our case, in order to reduce to a real Hamiltonian density it is necessary to consider as starting point two constraints that generate a complete set

$$(4.1) \quad \Sigma_1 = \pi_{0I} + \nabla^2 \pi_{2I} - m_2^2 \pi_{2I}, \quad \Sigma_2 = \pi_{0R} + \nabla^2 \pi_{2R} - m_1^2 \pi_{2R},$$

and their time evolution is

$$(4.2) \quad \dot{\Sigma}_1 = \int d^3x' \{ \Sigma_1, \mathcal{H}_{CR} \} = (-\nabla^2 + m_2^2) \Sigma_4 \quad \dot{\Sigma}_2 = (-\nabla^2 + m_1^2) \Sigma_3$$

where Σ_3 and Σ_4 are given by

$$(4.3) \quad \Sigma_3 = -\xi_R + \nabla^2 \phi_R - m_2^2 \phi_R, \quad \Sigma_4 = -\xi_I + \nabla^2 \phi_I - m_1^2 \phi_I.$$

In order to obtain the complete set we need to establish that these are the full set of constraints and then are closed.

The time evolution of the secondary constraints is

$$(4.4) \quad \dot{\Sigma}_3 = -\Sigma_2 \quad \dot{\Sigma}_4 = -\Sigma_1.$$

In this way the system is closed. The full algebra of constraints is

$$(4.5) \quad \{ \Sigma_a, \Sigma_b \} = \mathcal{D}_{ab} \delta(\vec{x} - \vec{x}')$$

and

$$(4.6) \quad \mathcal{D}_{ab} = \begin{pmatrix} 0 & 0 & 0 & -(m_1^2 - m_2^2) \\ 0 & 0 & -(m_1^2 - m_2^2) & 0 \\ 0 & (m_1^2 - m_2^2) & 0 & 0 \\ (m_1^2 - m_2^2) & 0 & 0 & 0 \end{pmatrix},$$

with determinant given by

$$(4.7) \quad \det(\mathcal{D}_{ab}) = (m_1^2 - m_2^2)^4$$

and the inverse matrix exist when $m_1 \neq m_2$ resulting

$$(4.8) \quad \mathcal{D}^{ab} = \begin{pmatrix} 0 & 0 & 0 & \beta^2 \\ 0 & 0 & \beta^2 & 0 \\ 0 & -\beta^2 & 0 & 0 \\ -\beta^2 & 0 & 0 & 0 \end{pmatrix},$$

with $\beta^2 = \frac{1}{(m_1^2 - m_2^2)}$. In conclusion since, we include 4 reality conditions as second constraints, in consequence the degrees of freedom of our theory change from 8 to 4. We must notice that the matrix (4.6) is invertible only in the case that $m_1 \neq m_2$. This means that for equal masses the reality conditions are not more second class constraints, and the theory will have several sectors [24, 11, 25, 26]. So, in this work we only consider the case $m_1 \neq m_2$.

The phase space that we have decided to use for convenience is given by $(\phi_R, \phi_I, \pi_{2R}, \pi_{2I})$. In order to set the new reduced theory, we establish the Dirac bracket

$$(4.9) \quad \{F_R(t, \vec{x}), G_R(t, \vec{x}_0)\}^{**} \equiv \{F_R(t, \vec{x}), G_R(t, \vec{x}_0)\}^* - \int d^3y \{F_R(t, \vec{x}), \Sigma_a(t, \vec{y})\}^* \mathcal{D}^{ab} \{\Sigma_b(t, \vec{y}), G_R(t, \vec{x}_0)\}^*$$

and the fundamental brackets are

$$(4.10) \quad \begin{aligned} \{\phi_R(t, \vec{x}), \pi_{2R}(t, \vec{x}_0)\}^{**} &= \frac{1}{(m_1^2 - m_2^2)} \delta^3(\vec{x} - \vec{x}_0), \\ \{\phi_I(t, \vec{x}), \pi_{2I}(t, \vec{x}_0)\}^{**} &= \frac{1}{(m_1^2 - m_2^2)} \delta^3(\vec{x} - \vec{x}_0), \end{aligned}$$

Though, in principle we can choose a different set of starting reality conditions this election is not arbitrary since in this case the cancelation of the imaginary part of the Hamiltonian density is quite clear. This statement implies that the phase space is established by the reality conditions (4.1) and (4.3) in order to express an inverse transformation using these conditions.

For this reason we emphasize that the phase space is $(\phi_R, \phi_I, \pi_{2R}, \pi_{2I})$. Applying strongly the conditions (4.1) and (4.3) in the density (3.21), we obtain the Hamiltonian density

$$(4.11) \quad \begin{aligned} \mathcal{H}_{CKG} &= \frac{(m_1^2 - m_2^2)}{2} \pi_{2R}^2 + \frac{m_2^2(m_1^2 - m_2^2)}{2} \phi_R^2 + \frac{(m_1^2 - m_2^2)}{2} (\nabla \phi_R)^2 \\ &+ \frac{(m_1^2 - m_2^2)}{2} \pi_{2I}^2 + \frac{m_1^2(m_1^2 - m_2^2)}{2} \phi_I^2 + \frac{(m_1^2 - m_2^2)}{2} (\nabla \phi_I)^2. \end{aligned}$$

The Lagrangian density (2.3) with the constraints and reality conditions is now

$$(4.12) \quad \begin{aligned} \mathcal{L}_{CKG} &= (\dot{\phi}\pi_0 + \dot{\xi}\pi_2)|_{cons} - \mathcal{H}_{CKG} = (m_1^2 - m_2^2) \dot{\phi}_R \pi_{2R} \\ &+ (m_1^2 - m_2^2) \dot{\phi}_I \pi_{2I} - \mathcal{H}_{CKG}. \end{aligned}$$

The expression (4.11) has a direct relationship with the Hamiltonian density of two real Klein-Gordon fields. The difference is only a contact transformation, given by

$$(4.13) \quad \begin{aligned} \sigma_R &= (m_1^2 - m_2^2)^{\frac{1}{2}} \phi_R, & p_R &= (m_1^2 - m_2^2)^{\frac{1}{2}} \pi_{2R}, \\ \sigma_I &= (m_1^2 - m_2^2)^{\frac{1}{2}} \phi_I, & p_I &= (m_1^2 - m_2^2)^{\frac{1}{2}} \pi_{2I}. \end{aligned}$$

Using this transformation in the Hamiltonian density (4.11) we get

$$(4.14) \quad \mathcal{H}_{KG} = \frac{1}{2} p_R^2 + \frac{m_2^2}{2} \sigma_R^2 + \frac{1}{2} (\nabla \sigma_R)^2 + \frac{1}{2} p_I^2 + \frac{m_1^2}{2} \sigma_I^2 + \frac{1}{2} (\nabla \sigma_I)^2.$$

The contact transformation (4.13) will be very useful in order to introduce the respective sources in the path integral formalism.

In next section we will explore in more detail the reality conditions (4.1) and (4.3) since by means of these structures we will include interaction potentials in this complex higher order model.

4.1. Interpretation of the Reality Conditions. Up to now, we have established that is possible to reduce a complex higher order system to a first order real system and it results to be Hermitian at the moment of quantizing. Reality conditions used as second class constraints [23] play a role fundamental in this reduction as well as to promote a contact transformation that allows to recognize the real system as the system of two real Klein-Gordon fields. However, this method is not the only possible since our starting point (4.1) was given by hand and not following a systematic procedure.

Taking this into account we observe that the particular map which relate the complex phase space $(\phi, \dot{\phi}, \pi_\phi, \pi_{\dot{\phi}})$ to the real phase space $(\psi_1, \pi_{\psi_1}, \psi_2, \pi_{\psi_2})$ is

$$(4.15) \quad \begin{aligned} \psi_1 &= \frac{1}{(m_1^2 - m_2^2)^{\frac{1}{2}}} (im_2^2 \phi - i(-\xi + \nabla^2 \phi)), \\ \psi_2 &= \frac{1}{(m_1^2 - m_2^2)^{\frac{1}{2}}} (m_1^2 \phi - (-\xi + \nabla^2 \phi)), \\ \pi_{\psi_1} &= i \frac{1}{(m_1^2 - m_2^2)^{\frac{1}{2}}} ((\pi_0 + \nabla^2 \pi_2) - m_1^2 \pi_2), \\ \pi_{\psi_2} &= \frac{1}{(m_1^2 - m_2^2)^{\frac{1}{2}}} ((\pi_0 + \nabla^2 \pi_2) - m_2^2 \pi_2). \end{aligned}$$

In order to show that the phase space $(\psi_1, \psi_2, \pi_{\psi_1}, \pi_{\psi_2})$ is real we assume that it is complex, resulting

$$\begin{aligned}
 (\psi_{1R} + i\psi_{1I}) &= \frac{1}{(m_1^2 - m_2^2)^{\frac{1}{2}}} [(m_1^2 - m_2^2)\phi_I - i\Sigma_3], \\
 (\psi_{2R} + i\psi_{2I}) &= \frac{1}{(m_1^2 - m_2^2)^{\frac{1}{2}}} [(m_1^2 - m_2^2)\phi_R - i\Sigma_4], \\
 (\pi_{\psi_{1R}} + i\pi_{\psi_{1I}}) &= \frac{1}{(m_1^2 - m_2^2)^{\frac{1}{2}}} [(m_1^2 - m_2^2)\pi_{2I} + i\Sigma_2], \\
 (4.16) \quad (\pi_{\psi_{2R}} + i\pi_{\psi_{2I}}) &= \frac{1}{(m_1^2 - m_2^2)^{\frac{1}{2}}} [(m_1^2 - m_2^2)\pi_{2R} + i\Sigma_1].
 \end{aligned}$$

In conclusion if the phase space is complex, the imaginary part is proportional to the reality conditions or constraints (4.1) and (4.3) and then the phase space is real implementing these conditions.

The reality conditions obtained from the complex mapping, that generates a real phase space, results in reality conditions which are no minimal expressions, since

$$\begin{aligned}
 (m_1^2 - m_2^2)\phi^* &= (m_1^2 + m_2^2)\phi + 2(\xi - \nabla^2\phi), \\
 (m_1^2 - m_2^2)\pi_2^* &= -(m_1^2 + m_2^2)\pi_2 + 2(\pi_0 + \nabla^2\pi_2), \\
 (m_1^2 - m_2^2)(\pi_0^* + \nabla^2\pi_2^*) &= (m_1^2 + m_2^2)(\pi_0 + \nabla^2\pi_2) - 2m_1^2m_2^2\pi_2, \\
 (4.17) \quad (m_1^2 - m_2^2)(-\xi^* + \nabla^2\phi^*) &= -(m_1^2 + m_2^2)(-\xi + \nabla^2\phi) + 2m_1^2m_2^2\phi_2
 \end{aligned}$$

and by separating in real and imaginary parts

$$\begin{aligned}
 \Sigma_3 + i\Sigma_4 &= 0, \quad \Sigma_2 + i\Sigma_1 = 0, \\
 (4.18) \quad m_2^2\Sigma_2 + im_1^2\Sigma_4 &= 0, \quad m_1^2\Sigma_3 + im_2^2\Sigma_4 = 0.
 \end{aligned}$$

The conditions (4.17) can't be used as constraints [15] since the conjugated variables in our system aren't dynamic variables unless we consider the components as real independent fields in that case (4.1) and (4.3) have a similar complex dynamics to our proposal.

The relationship between the method described with second class constraints or reality conditions and the method of a complex canonical transformation is

$$\begin{aligned}
 \sigma_R &= \psi_2, \quad \sigma_I = \psi_1, \\
 (4.19) \quad p_R &= \pi_2, \quad p_I = \pi_1.
 \end{aligned}$$

Summarizing, starting from the complex Bernard-Duncan model with 12 real degrees of freedom, we reduce it to four real degrees of freedom. For that we use four constraints that appears from the definition of the momenta Eq. (3.8) and four reality conditions. In this way the reality conditions generate a real first order theory that is directly related to complex higher order theory. It is important to highlight that the reality conditions are second class constraints whose Poisson bracket is a constant matrix. So at the quantum level we don't have problems to implement

these brackets. This description is incomplete since this model has neither the self-interactions between the fields nor the interaction with external fields. In order to introduce these effects in this model, we will aggregate self-interaction potentials whose application of the reality conditions result in consistent potentials into the reduced space.

5. THE INTERACTION POTENTIALS

With the purpose of including interactions into the model we take into account the following criteria: i) The interactions must be real quantities one's we apply the reality condition and constraints. ii) In principle it is possible to include real interactions that dependent of momenta, we don't take into account this possibility. This possibility modifies the definition of the momenta and it can result in a non-Lorentz-invariant theory. iii) By consistency of the theory we require that the interacting Hamiltonian does not generate new constraints, to implement this condition we select a set of interactions that commute with the reality conditions Σ_1 and Σ_2 . In this way, it is possible to choose interaction potentials that are exclusively dependent of the fields and in addition automatically to commute with the reality conditions Σ_3 and Σ_4 . This criterion assumes that the time evolution of the constraints is not modified by the interaction terms.

5.1. Selection Criteria of the Interaction Potentials. In order to study these criteria we going to consider an example in a way that it will be possible to extend to other interaction potentials. To select this interaction potential we take into account that applying the reality conditions this potential results in a real Lorentz invariant term. In order to find such expression, we consider the conjugate field ϕ^* but since it is not a variable of the system, we replace the conjugate expression using the reality conditions. In this way, the reality conditions allow to find consistent real interaction potentials at the reduced space.

Now, it is possible to define a bar field inside the extended space that collapses in a conjugate field inside the reduced space, resulting

$$(5.1) \quad \bar{\phi} = \frac{(m_1^2 + m_2^2)}{(m_1^2 - m_2^2)}\phi - \frac{2}{(m_1^2 - m_2^2)}(-\xi + \nabla^2\phi),$$

whenever $m_1 \neq m_2$ and different from zero. The expression of (5.1) is $\bar{\phi} \neq \phi^*$ on the extended space in general, but it can be reduced to the conjugate field inside the reduced space, resulting

$$(5.2) \quad \phi^* = \bar{\phi}|_{rec} = \phi_R - i\phi_I.$$

This reduced space is gotten of applying the reality conditions Σ_3 and Σ_4 on the extended space which we will denote by $|_{rec}$.

The last statement suggests that the selection of interaction potentials inside the extended space is dependent of the reality conditions in such a way that by

restricting the phase space the interaction potentials don't leave the reduced space. The components of field in terms of ϕ and his conjugate $\bar{\phi}$ are

$$(5.3) \quad \phi_R = \frac{1}{2}(\phi + \bar{\phi})|_{cre}, \quad \phi_I = \frac{1}{2i}(\phi - \bar{\phi})|_{cre}$$

that allow us to introduce the possible interaction potentials

$$(5.4) \quad \begin{aligned} U_{int}^1(\phi, \xi) &= \int d^3x \frac{g_1}{4!(m_1^2 - m_2^2)^2} [m_1^2\phi + (\xi - \nabla^2\phi)]^4, \\ U_{int}^2(\phi, \xi) &= \int d^3x \frac{g_2}{4!(m_1^2 - m_2^2)^2} [m_2^2\phi + (\xi - \nabla^2\phi)]^4, \\ U_{int}^3(\phi, \xi) &= \int d^3x \frac{g_3}{4!(m_1^2 - m_2^2)^2} [m_1^2\phi + (\xi - \nabla^2\phi)]^2 [m_2^2\phi + (\xi - \nabla^2\phi)]^2. \end{aligned}$$

Including these expressions we obtain an interacting Hamiltonian density that is called total Hamiltonian density and if we commute it with the reality conditions we obtain proportional elements to the reality conditions since every potential is real.

The interaction potentials (5.4) applying the reality conditions (4.3) are

$$(5.5) \quad \begin{aligned} U_{int}^1|_{cre} &= \int d^3x \frac{g_1}{4!} \psi_2^4, & U_{int}^2|_{cre} &= \int d^3x \frac{g_2}{4!} \psi_1^4, \\ U_{int}^3|_{cre} &= \int d^3x \frac{g_3}{4!} \psi_2^2 \psi_1^2. \end{aligned}$$

The last procedure establishes a way of introducing self-interactions which can be applied in a systematic form. In the next section we shall explore another kind of interaction that is possible to introduce and we will consider the path integral quantization of the model.

5.1.1. Some Interaction Potentials. Following the last procedure, we can consider some different interaction potentials that have the same characteristic in common. From the Hamiltonian formalism we write the real and imaginary parts in terms of the fields (5.3), resulting

$$(5.6) \quad \begin{aligned} U_{int}^4(\phi, \xi) &= \int d^3x \frac{-g_4}{(m_1^2 - m_2^2)^{\frac{3}{2}}} (m_1^2\phi + \xi - \nabla^2\phi)(m_2^2\phi + \xi - \nabla^2\phi)^2, \\ U_{int}^5(\phi, \xi) &= \int d^3x \frac{g_5}{(m_1^2 - m_2^2)^{\frac{3}{2}}} (m_1^2\phi + \xi - \nabla^2\phi)^3, \\ U_{int}^6(\phi, \xi) &= \int d^3x \frac{-ig_6}{(m_1^2 - m_2^2)^{\frac{3}{2}}} (m_2^2\phi + \xi - \nabla^2\phi)^3, \\ U_{int}^7(\phi, \xi) &= \int d^3x \frac{-ig_7}{(m_1^2 - m_2^2)^{\frac{3}{2}}} (m_2^2\phi + \xi - \nabla^2\phi)(m_1^2\phi + \xi - \nabla^2\phi)^2, \end{aligned}$$

where by applying the reality conditions (4.3), we obtain

$$(5.7) \quad \begin{aligned} U_{int}^4|_{cre} &= g_4 \psi_2 \psi_1^2, & U_{int}^5|_{cre} &= g_5 \psi_2^3, \\ U_{int}^6|_{cre} &= g_6 \psi_1^3, & U_{int}^7|_{cre} &= \psi_1 \psi_2^2. \end{aligned}$$

It is important to emphasize that the reality conditions or second class constraints are a closed set under the time evolution. We showed it in the equations (4.16) where was exhibited that the imaginary part of the fields ψ 's vanish module the reality conditions (4.1), (4.3). Now, if we include the potentials (5.5) and (5.6) to the Hamiltonian density (3.21), the evolution of the reality conditions is not modified by interactions. This will be helpful in order to build the path integral in such a way that the sources are consistent with the theory. Furthermore, considering the dependence between the fields due to the constraints and reality conditions we will have to take into account the dependence between the sources.

6. PATH INTEGRAL AND COMPLEX SOURCES

In order to establish the path integral and to introduce the appropriate sources of the fields, we have to analyze the classical properties of these sources. These properties express a relationship between the sources in such way that the imaginary terms are canceled in the Hamiltonian density and in this form the Cauchy-Riemann equations will be preserved. With these principles, the path integral formulation with currents is quite manageable.

6.1. Equations of Motion with currents. Let us consider a free complex Hamiltonian density which include the currents of fields and momenta

$$(6.1) \quad \mathcal{H}_S = \mathcal{H} - J\phi - K\pi_0 - L\eta - M\pi_1 - N\xi - O\pi_2.$$

Note that we have included every kind of currents that belong to the complex Bernard-Duncan model. Using the Cauchy-Riemann conditions for the complex

Hamiltonian density (6.1), together with (2.12), we obtain the equations of motion

$$\begin{aligned}
(6.2) \quad \dot{\phi}_R &= \frac{\partial \mathcal{H}_{SR}}{\partial \pi_{0R}} = 2\pi_{2R} - \eta_R - K_R, & \dot{\phi}_I &= -\frac{\partial \mathcal{H}_{SR}}{\partial \pi_{0I}} = 2\pi_{2I} - \eta_I - K_I, \\
\dot{\pi}_{0R} &= \frac{\partial \mathcal{H}_{SR}}{\partial \phi_R} = \nabla^2(\nabla^2 \phi_R) - (m_1^2 + m_2^2)\nabla^2 \phi_R + m_1^2 m_2^2 \phi_R - J_R, \\
\dot{\pi}_{0I} &= \frac{\partial \mathcal{H}_{SR}}{\partial \phi_I} = -\nabla^2(\nabla^2 \phi_I) + (m_1^2 + m_2^2)\nabla^2 \phi_I - m_1^2 m_2^2 \phi_I + J_I, \\
\dot{\eta}_R &= \frac{\partial \mathcal{H}_{SR}}{\partial \pi_{1R}} = \xi_R - M_R, & \dot{\eta}_I &= -\frac{\partial \mathcal{H}_{SR}}{\partial \pi_{1I}} = \xi_I - M_I, \\
\dot{\pi}_{1R} &= \frac{\partial \mathcal{H}_{SR}}{\partial \eta_R} = 2\nabla^2 \pi_{2R} - (m_1^2 + m_2^2)\pi_{2R} + (m_1^2 + m_2^2)\eta_R \\
&\quad - 2\nabla^2 \eta_R - \pi_{0R} - L_R, \\
\dot{\pi}_{1I} &= \frac{\partial \mathcal{H}_{SR}}{\partial \eta_I} = -2\nabla^2 \pi_{2I} + (m_1^2 + m_2^2)\pi_{2I} - (m_1^2 + m_2^2)\eta_I \\
&\quad + 2\nabla^2 \eta_I + \pi_{0I} + L_I, \\
\dot{\xi}_R &= \frac{\partial \mathcal{H}_{SR}}{\partial \pi_{2R}} = 2\pi_{0R} + 2\nabla^2 \eta_R - (m_1^2 + m_2^2)\eta_R - O_R, \\
\dot{\xi}_I &= -\frac{\partial \mathcal{H}_{SR}}{\partial \pi_{2I}} = 2\pi_{0I} + 2\nabla^2 \eta_I - (m_1^2 + m_2^2)\eta_I - O_I, \\
\dot{\pi}_{2R} &= \frac{\partial \mathcal{H}_{SR}}{\partial \xi_R} = \pi_{1R} - \xi_R - N_R, & \dot{\pi}_{2I} &= \frac{\partial \mathcal{H}_{SR}}{\partial \xi_I} = -\pi_{1I} + \xi_I + N_I.
\end{aligned}$$

Using the equations of motion (6.2), the four constraints resulting of the momenta (3.8) and, the four reality conditions (4.1) and (4.3), we obtain that the currents are related by

$$\begin{aligned}
(6.3) \quad J_R &= (-\nabla^2 + m_1^2)N_R, & J_I &= (-\nabla^2 + m_2^2)N_I \\
L_R + O_R &= (\nabla^2 - m_2^2)K_R, & L_I + O_I &= (\nabla^2 - m_1^2)K_I.
\end{aligned}$$

From the expression (6.1) the currents are independent quantities into the extended space, but if we consider the reduced space defined by the constraints and the reality conditions. The currents aren't independent and we obtain relationships between them (6.3). In our procedure, the relationships are established by the constraints and reality conditions and the total derivative plays a fundamental role in order to define the fields and the currents.

6.2. Path Integral. In this section, we will be concerned with establishing the complex higher order theory in terms of the path integral formalism, in order to get a consistent quantization that includes interaction potentials. To introduce the path integral, we shall use an integration measure that considers every constraint and reality condition. Taking as starting point the complex Hamiltonian density with complex currents (6.1), we will build the integration measure on the path integral following the Senjanovic's method [27]. However, it is necessary to add independently the classical relations between the currents, since they are classical

fields and are not quantized in the usual description. It is important to mention that the source terms into the Hamiltonian density are complex quantities that obey the equations of Cauchy-Riemann for multiple variables, that are handled in the expressions above (6.3), in such a way that the imaginary part of the Hamiltonian density is zero including every constraint and reality condition.

To begin with, we study the path integral with currents that describe the annihilation and creation process. In order to include the currents, we choose the respective complex fields, ϕ , η , ξ and their respective momenta π_0 , π_1 , π_2 . So, to introduce the measure of integration, we consider in the path integral that the real and imaginary parts are independent, and we include as functional Dirac's deltas every second class constraint resulting of the momenta (3.8) together with the reality conditions (4.1) and (4.3).

To apply the Senjanovic's method we define the set of constraints as

$$(6.4) \quad \Omega_i = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$$

that allows to establish the respective determinant of the Poisson brackets, resulting

$$(6.5) \quad \{\Omega_i, \Omega_j\} = \mathcal{G}_{ij} \delta(\vec{x} - \vec{x}'),$$

with

$$(6.6) \quad \det(\mathcal{G}_{ij}) = (m_1^2 - m_2^2)^4.$$

The integration measure on the path integral is

$$(6.7) \quad \mathcal{D}\mu = \mathcal{D}\Theta_R^a \mathcal{D}\Pi_{aR} \mathcal{D}\Theta_I^a \mathcal{D}\Pi_{aI} \det |\{\Omega_i, \Omega_j\}| \prod_i^8 \delta(\Omega_i)$$

where \prod_i regards a product of Dirac deltas on each constraint.

The currents in terms of this notation are

$$(6.8) \quad \begin{aligned} \mathcal{J}_{aA} &= (J_R, L_R, N_R, J_I, L_I, N_I) \\ \mathcal{K}_B^b &= (K_R, M_R, O_R, K_I, M_I, O_I) \end{aligned}$$

with $a = J, L, N$, $b = K, M, O$ and $A, B = R, I$. The generating functional using the above elements is

$$(6.9) \quad \begin{aligned} Z = \int \mathcal{D}\mu \exp[i \int d^4x (\dot{\Theta}_R^a \Pi_{aR} - \dot{\Theta}_I^a \Pi_{aI} - \mathcal{H}_R + \Theta_R^a \mathcal{J}_{aR} - \Theta_I^a \mathcal{J}_{aI} \\ + \Pi_{aR} \mathcal{K}_R^a - \Pi_{aI} \mathcal{K}_I^a) + i(\dot{\Theta}_R^a \Pi_{aI} + \dot{\Theta}_I^a \Pi_{aR} - \mathcal{H}_I + \Theta_R^a \mathcal{J}_{aI} + \Theta_I^a \mathcal{J}_{aR} \\ + \Pi_{aR} \mathcal{K}_I^a + \Pi_{aI} \mathcal{K}_R^a)], \end{aligned}$$

where \mathcal{H}_R and \mathcal{H}_I are given in (2.13) and (2.14) respectively and we use the notation given in (3.2), (3.3). We must notice that the determinant of the second class constraints only contribute to the generating functional with a constant factor proportional to the difference of the masses of the scalar fields. This factor is

not important in the case of different masses, but perhaps is quite relevant in the limit of equal masses.

Using the Dirac's delta functional in (6.9) we integrate over the fields $(\eta_R, \eta_I, \xi_R, \xi_I, \pi_{0R}, \pi_{0I}, \pi_{1R}, \pi_{1I})$. From this the path integral is reduced to

$$\begin{aligned}
 Z_R = \int \mathcal{D}\phi_R \mathcal{D}\phi_I \mathcal{D}\pi_{2R} \mathcal{D}\pi_{2I} \exp\{i \int d^4x & [(m_1^2 - m_2^2)\dot{\phi}_R \pi_{2R} + (m_1^2 - m_2^2) \\
 & \dot{\phi}_I \pi_{2I} - \mathcal{H}_{CKG} + \phi_R(\nabla^2 N_R - m_2^2 N_R + J_R) + \phi_I(-\nabla^2 N_I + m_1^2 N_I - J_I) \\
 & + \pi_{2R}(-\nabla^2 K_R + m_1^2 K_R + L_R + O_R) + \pi_{2I}(\nabla^2 K_I - m_2^2 K_I - L_I - O_I)] \\
 & + i[\phi_R(\nabla^2 N_I - m_2^2 N_I + J_I) + \phi_I(\nabla^2 N_R - m_1^2 N_R + J_R) \\
 & + \pi_{2I}(-\nabla^2 K_R + m_2^2 K_R + L_R + O_R) \\
 & + \pi_{2R}(-\nabla^2 K_I + m_1^2 K_I + L_I + O_I)]\}.
 \end{aligned}
 \tag{6.10}$$

and since the Hamiltonian density is still complex, this expression does not include completely the final map to the real space.

Up to this point, we have the path integral without constraints. The benefit of preserving the imaginary part of the Hamiltonian density, is that in the imaginary part of (6.10), we obtain the equations (6.3) as a consequence of the measure of the path integral. In conclusion, the imaginary part, that could generate ghosts, disappears from the path integral using the classical relationship of the currents (6.3). In this way, we obtain

$$\begin{aligned}
 Z_R = \int \mathcal{D}\phi_R \mathcal{D}\phi_I \mathcal{D}\pi_{2R} \mathcal{D}\pi_{2I} \exp\{i \int d^4x & [(m_1^2 - m_2^2)\dot{\phi}_R \pi_{2R} \\
 & + (m_1^2 - m_2^2)\dot{\phi}_I \pi_{2I} - \mathcal{H}_{CKG} + (m_1^2 - m_2^2)\phi_R N_R + (m_1^2 - m_2^2)\phi_I N_I \\
 & + (m_1^2 - m_2^2)\pi_{2R} K_R + (m_1^2 - m_2^2)\pi_{2I} K_I]\}.
 \end{aligned}
 \tag{6.11}$$

Finally, we can apply the contact transformation

$$\begin{aligned}
 \sigma_R &= (m_1^2 - m_2^2)^{\frac{1}{2}} \phi_R, & p_R &= (m_1^2 - m_2^2)^{\frac{1}{2}} \pi_{2R}, \\
 \sigma_I &= (m_1^2 - m_2^2)^{\frac{1}{2}} \phi_I, & p_I &= (m_1^2 - m_2^2)^{\frac{1}{2}} \pi_{2I}, \\
 \aleph_R &= (m_1^2 - m_2^2)^{\frac{1}{2}} N_R, & \aleph_I &= (m_1^2 - m_2^2)^{\frac{1}{2}} N_I, \\
 \mathbb{k}_R &= (m_1^2 - m_2^2)^{\frac{1}{2}} K_R, & \mathbb{k}_I &= (m_1^2 - m_2^2)^{\frac{1}{2}} K_I,
 \end{aligned}
 \tag{6.12}$$

and the Jacobian of this transformation is

$$|J| = \frac{1}{(m_1^2 - m_2^2)^2}, \quad d\phi_R d\pi_{2R} d\phi_I d\pi_{2I} = |J| d\sigma_R dp_R d\sigma_I dp_I.
 \tag{6.13}$$

The expressions (6.13) transform the path integral (6.11) into

$$\begin{aligned}
 Z_{KG} = S \int \mathcal{D}\sigma_R \mathcal{D}\sigma_I \mathcal{D}p_R \mathcal{D}p_I \exp\{i \int d^4x & (\dot{\sigma}_R p_R + \dot{\sigma}_I p_I \\
 & - \mathcal{H}_{KG}(\sigma_R, \sigma_I, p_R, p_I) + \sigma_R \aleph_R + \sigma_I \aleph_I + p_R \mathbb{k}_R + p_I \mathbb{k}_I)\}
 \end{aligned}
 \tag{6.14}$$

where \mathcal{H}_{KG} is the expression (4.14). In expression (6.14), S is a constant quantity that results from the determinant (6.6) and the jacobian expression (6.13) resulting

from the contact transformation. It will allow to define the generating functional of interactions using the free generating functional.

6.2.1. *Generating Functional of Interactions.* Using the free generating functional (6.14) it is possible to build the generating functional with interactions and we obtain

$$(6.15) \quad Z_{int} = \int \mathcal{D}\sigma_R \mathcal{D}\sigma_I \mathcal{D}p_R \mathcal{D}p_I \exp\{i \int d^4x (\dot{\sigma}_R p_R + \dot{\sigma}_I p_I - \mathcal{H}_{CKG} - \frac{g_1}{4!} \sigma_R^4 - \frac{g_2}{4!} \sigma_I^4 - \frac{g_3}{4!} \sigma_R^2 \sigma_I^2 + \sigma_R \aleph_R + \sigma_I \aleph_I + p_R \mathbb{K}_R + p_I \mathbb{K}_I)\}$$

that in terms of the free generating functional are

$$(6.16) \quad Z_{int1} = \exp\{-\frac{ig_1}{4!} \int d^4x (\frac{\delta}{i\delta \aleph_R})^4\} Z_{KG},$$

$$(6.17) \quad Z_{int2} = \exp\{-\frac{ig_2}{4!} \int d^4x (\frac{\delta}{i\delta \aleph_I})^4\} Z_{KG},$$

$$(6.18) \quad Z_{int3} = \exp\{-\frac{ig_3}{4!} \int d^4x (\frac{\delta}{i\delta \aleph_R})^2 (\frac{\delta}{i\delta \aleph_I})^2\} Z_{KG},$$

and it shows that is possible to introduce interactions in the high order derivative theory whereas these interactions are compatible with the constraints and the reality conditions. In this way, our procedure allows to quantize the complex higher order theory (2.3), with the interactions (5.4). In next section, we will apply this method to the Schwinger model in order to check for an explicit example how it works.

7. USING THE METHOD

In order to explore the scope of the method that includes the boundary conditions $(\phi, \ddot{\phi})$, we analyze a concrete example that shows limit cases of our description. Between all the possible examples, we have the Schwinger model that is a good starting point since it has been explored exhaustively and is a very important model, if we want to study higher order time derivative theories. In this electrodynamics into two dimensions, we find a phenomenon known as Bosonization that is easily viewed using the higher order time derivative theories [17]. In this section, we apply the method here described, starting from the general description of the 2-dimensional electrodynamics to a real higher order theory that is a particular case of our complex model.

7.1. The Schwinger model. In this section, we apply the above method to the Schwinger model, and we compare with a previous procedure [17].

Consider the Schwinger model that is a formulation of the massless electrodynamics in 1 + 1 dimensions. The Lagrangian density of departure is

$$(7.1) \quad \mathcal{L}_{ED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i\gamma^\mu(\partial_\mu - ieA_\mu)]\psi,$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This electromagnetic Lagrangian density is coupled to a Dirac Lagrangian density with zero mass. On the classical level it is well known that the vector current is conserved, but we must also consider the chiral current that results classically to be also a conserved quantity, since the electron has zero mass. But the focus is on the respective quantized model. In this case the chiral current is not more a conserved quantity resulting in a breaking of the classical symmetry. This phenomenon is known as chiral anomaly and its consequences can be studied by means of a non local transformation of (7.1) that is given by

$$(7.2) \quad A_\mu = -\frac{1}{e}\varepsilon_{\mu\nu}\partial^\nu\varphi + \frac{1}{e}\partial_\mu\eta, \quad \psi = \exp(i\gamma^5\varphi + i\eta)\kappa, \\ \bar{\psi} = \bar{\kappa}\exp(i\gamma^5\varphi - i\eta).$$

with two-dimensional Dirac matrices

$$(7.3) \quad [\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu}, \quad \gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\gamma_\nu, \\ \epsilon_{01} = -\epsilon_{10}.$$

Using the expressions (7.2) we get

$$(7.4) \quad F_{\mu\nu} = \frac{1}{e}\varepsilon_{\mu\nu}\square\varphi, \quad \square\eta = 0.$$

and the gauge transformation is expressed in terms of the field η

$$(7.5) \quad A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x) = -\frac{1}{e}\varepsilon_{\mu\nu}\partial^\nu\phi + \frac{1}{e}\partial_\mu(\eta + e\Lambda) = -\frac{1}{e}\varepsilon_{\mu\nu}\partial^\nu\phi + \frac{1}{e}\partial_\mu\eta',$$

implying that this field doesn't appears in the Lagrangian density. The classical Lagrangian density using the transformation (7.2) is

$$(7.6) \quad \mathcal{L} = \frac{1}{2e^2}(\square\varphi)^2 + \bar{\kappa}i\gamma^\mu\partial_\mu\kappa,$$

where it has been decouple the bosonic part given by the field φ from the fermionic part given by the field κ . In order to realize the quantization, it is used the path integral in which the non local transformation is applied (7.2), implying the appearance of the chiral anomaly in the associated Faddeev-Popov Jacobian. To cancel this anomaly the total Lagrangian density acquires a new term. Our starting point is precisely this full effective action giving place to

$$(7.7) \quad S_0 = \int d^4x \frac{1}{2}[-(\square\varphi)^2 + m_1^2\partial_\mu\varphi\partial^\mu\varphi]$$

with the field φ real. An important fact to bear in mind is that the Schwinger model is a particular case of the Bernard-Duncan model (2.1) with $m_2^2 = 0$ and

$m_1^2 = \frac{e^2}{\pi}$. The next step is to realize the complexification as we described previously including the total derivative, with the result

$$(7.8) \quad S = \int d^4x \frac{1}{2} [(\Box\phi)^2 + m_1^2 \partial_\mu \phi \partial^\mu \phi] + \partial_\mu \phi (\partial^\mu \Box\phi)$$

and applying a variation of the action in order to obtain the momenta, we have

$$(7.9) \quad \pi_0 = \phi^{(3)} - 2\nabla^2 \dot{\phi} + m_1^2 \dot{\phi}, \quad \pi_1 = 0, \quad \pi_2 = \dot{\phi}.$$

With these momenta we can obtain the respective Hamiltonian density

$$(7.10) \quad \mathcal{H}_S = \pi_1 \xi + \pi_0 \pi_2 + 2\pi_2 \nabla^2 \eta - m_1^2 \eta \pi_2 - \frac{1}{2} \xi^2 - \frac{1}{2} (\nabla^2 \phi)^2 + \xi \nabla^2 \phi \\ - \frac{m_1^2}{2} \eta^2 + \frac{m_1^2}{2} \nabla \phi \cdot \nabla \phi - \eta \nabla^2 \eta + m_1^2 \eta^2 - \xi \nabla^2 \phi + (\nabla^2 \phi)^2 \\ + \pi_0 \pi_2 - \eta \pi_0$$

and, we obtain the respective constraints

$$(7.11) \quad \gamma_1 = \pi_{1R}, \quad \gamma_2 = \pi_{1I}, \\ \gamma_3 = \pi_{2R} - \eta_R, \quad \gamma_4 = \pi_{2I} - \eta_I.$$

These restrictions are second class constraints. Using these constraints we define the respective Dirac brackets

$$(7.12) \quad \{\phi_R(t, \vec{x}), \pi_{0R}(t, \vec{x}_0)\}^* = \delta^3(\vec{x} - \vec{x}_0) \\ \{\phi_I(t, \vec{x}), \pi_{0I}(t, \vec{x}_0)\}^* = -\delta^3(\vec{x} - \vec{x}_0) \\ \{\eta_R(t, \vec{x}), \xi_R(t, \vec{x}_0)\}^* = -\delta^3(\vec{x} - \vec{x}_0) \\ \{\eta_I(t, \vec{x}), \xi_I(t, \vec{x}_0)\}^* = \delta^3(\vec{x} - \vec{x}_0) \\ \{\xi_R(t, \vec{x}), \pi_{2R}(t, \vec{x}_0)\}^* = \delta^3(\vec{x} - \vec{x}_0) \\ \{\xi_I(t, \vec{x}), \pi_{2I}(t, \vec{x}_0)\}^* = -\delta^3(\vec{x} - \vec{x}_0).$$

The second class constraints are applied strongly to the Hamiltonian density ones the Dirac brackets are imposed. The resulting Hamiltonian density is

$$(7.13) \quad \mathcal{H}_C = \pi_0 \pi_2 - \frac{1}{2} \xi^2 - \frac{m_1^2}{2} \pi_2^2 + \frac{1}{2} (\nabla^2 \phi)^2 + \frac{m_1^2}{2} (\nabla \phi)^2 + \pi_2 \nabla^2 \pi_2.$$

The following step is to impose the reality conditions in order to reduce the complex space to real space and check if the time evolution does not generate another constraint. In this case, the complete set of reality conditions is

$$(7.14) \quad \Sigma_1 = \pi_{0I} + \nabla^2 \pi_{2I}, \quad \Sigma_4 = -\xi_I + \nabla^2 \phi_I - m_1^2 \phi_I \\ \Sigma_2 = \pi_{0R} + \nabla^2 \pi_{2R} - m_1^2 \pi_{2R}, \quad \Sigma_3 = -\xi_R + \nabla^2 \phi_R$$

where Σ_1 with Σ_2 are two arbitrary constraints and Σ_3 with Σ_4 are consequence of the time evolution of the first two constraints. These second class constraints imply the next Dirac brackets

$$(7.15) \quad \begin{aligned} \{\phi_R(t, \vec{x}), \pi_{2R}(t, \vec{x}_0)\}^{**} &= \frac{1}{m_1^2} \delta^3(\vec{x} - \vec{x}_0), \\ \{\phi_I(t, \vec{x}), \pi_{2I}(t, \vec{x}_0)\}^{**} &= \frac{1}{m_1^2} \delta^3(\vec{x} - \vec{x}_0). \end{aligned}$$

The real Hamiltonian density is

$$(7.16) \quad \mathcal{H}_{CKG} = \frac{m_1^2}{2} \pi_{2I}^2 + \frac{m_1^2}{2} (\nabla \phi_I)^2 + \frac{m_1^4}{2} \phi_I^2 + \frac{m_1^2}{2} \pi_{2R}^2 + \frac{m_1^2}{2} (\nabla \phi_R)^2$$

This Hamiltonian density isn't a higher order time derivative theory and is a real quantity. Now, it is possible to define a contact transformation

$$(7.17) \quad \sigma_R = m_1 \phi_R, \quad p_R = m_1 \pi_{2R}, \quad \sigma_I = m_1 \phi_I, \quad p_I = m_1 \pi_{2I}.$$

Using the above expressions the new Dirac brackets are

$$(7.18) \quad \begin{aligned} \{\sigma_R(t, \vec{x}), p_R(t, \vec{x}_0)\}^{**} &= \delta^3(\vec{x} - \vec{x}_0), \\ \{\sigma_I(t, \vec{x}), p_I(t, \vec{x}_0)\}^{**} &= \delta^3(\vec{x} - \vec{x}_0), \end{aligned}$$

and the Hamiltonian density is

$$(7.19) \quad \mathcal{H}_{KG} = \frac{1}{2} p_I^2 + \frac{1}{2} (\nabla \sigma_I)^2 + \frac{m_1^2}{2} \sigma_I^2 + \frac{1}{2} p_R^2 + \frac{1}{2} (\nabla \sigma_R)^2.$$

In this way, we see the relationship between our real Hamiltonian density (7.19) with our complex higher order time derivative Hamiltonian density (7.13). This relationship shows the complex Schwinger model with the border conditions $(\phi, \dot{\phi})$ is linked to the Hamiltonian density of two real Klein-Gordon fields, a massless field and other massive excitation. This result is well known [28] and in this way our procedure reproduces the correct result. Our method is also applicable to another boundary condition in such a way that it is possible to fix $(\phi, \dot{\phi})$ without the total derivative.

8. CONCLUSIONS

In this work, it was introduced a method that makes it possible to map from a complex higher order derivative theory with interaction potentials, the complex interacting Bernard-Duncan model, to an interacting theory of two real Klein-Gordon fields. This complex extension to the higher order derivative theory is a consequence of thinking in a more general theory with a complex structure that is possible to quantize using the reality conditions and avoiding problems as unbounded energy and states of negative norm. The complex theory allows to gain more flexibility in order to establish a different concept of hermiticity, by means of reality condition, resulting of a complex classical mechanics to describe a higher order derivative theory. The key point was to establish a complex structure into the Hamiltonian

formalism using Ostrogradsky method and to develop a mapping mechanism to a real space. The basic idea is to use the reality conditions [22], corresponding to this complex theory, as second class constraint and to aggregate interaction potentials that are mapped to real quantities applying these conditions. It is not easy to set the interaction potentials because they must not generate additional conditions that constrain the degrees of freedom when the temporal evolution is done. Moreover we showed that is possible to set up a complex description using the classical mechanics, stated by Ostrogradsky, consistent with Hamilton's equations and Cauchy-Riemann equations. The information that is given by this complex description is predetermined in such a way that the complex structure of multi variables is respected and in consequence the Cauchy-Riemann equations are satisfied. Another step in the method was to raise the derivative order of the theory in such way to select the appropriated boundary conditions. The original order of the theory is recovered through second class constraints using the Dirac's theory of constraints [18, 19]. However, this complex higher order derivative theory has double degrees of freedom when compared to the real Bernard-Duncan theory. In order to reduce this extra degrees of freedom, we directly introduce two constraints that evolve in another two constraints resulting in a complete set of second class that reduces this theory to a real description. These constraints are the reality conditions.

From this description, the quantization is possible using the path integral with the Senjanovic's method [27] developed to quantize a theory with second class constraints. Using complex currents and developing a formalism that includes them separating in components the constraints are applied and the relationship between higher order fields is exhibited resulting from a complex structure and reality conditions. The complex currents are not independent in this higher order derivative model if reality conditions are established. This is consistently with Ostrodrasky's method [20]. As a final point on the path integral applied to the complex Bernard-Duncan model with currents, we conclude that it can reduce to quantize two Klein-Gordon fields with currents using the path integral and applying constraints and taking into account a contact transformation. Furthermore, the cancelation of the imaginary part of the path integral generates the relations between the currents which are also obtained classically by means of Hamiltonian equations of motion including currents.

To include interaction potentials, we considered quantities in the Hamiltonian density that do not generate new constraints and they must preserve a closed set of constraints when the temporal evolution is established (5.4). These interaction potentials are generated by currents either in the complex space or real space using the reality conditions. The interaction potentials be attached to the Hamiltonian density result in a renormalizable theory that is defined into the reduced space (5.5)

[16]. Finally, we used that method in the Schwinger model [17] obtaining consistent results.

REFERENCES

- [1] A. Pais and G. E. Uhlenbeck, On Field with Non- Localized Action, Phys. Rev. **79** (1950), 145.
- [2] C. Bernard and A. Duncan, Lorentz covariance and Matthews's theorem for derivative-coupled field theories, Phys. Rev. **D 11** (1975), 848.
- [3] C. Grosse-Knetter, Effective Lagrangians with higher derivatives and equations of motion, Phys. Rev. **D 49** (1994), 6709.
- [4] D. A. Eliezer and R. P. Woodard, The Problem of Nonlocality In String Theory, Nucl. Phys. **B 325** (1989), 389.
- [5] S. W. Hawking and T. Hertog, Living with ghosts, Phys. Rev. **D 65**, 103515(2002).
- [6] T-C. Cheng, P-M. Ho, M-C. Yeh, Perturbative approach to higher derivative and nonlocal theories, Nucl. Phys. **B 625** (2002), 151.
- [7] T. Nakamura and S. Hamamoto, Higher Derivatives and Canonical Formalisms, Prog. Theor. Phys. **95** (1996), 469.
- [8] K. Andrzejewski, J. Gonera, and P. Mařlanka, Euclidean Path Integral and Higher-Derivative Theories, Prog. Theor. Phys. **125** (2011), 247.
- [9] I. Antoniadis, E. Dudas and D. M. Ghilencea, Living with ghosts and their radiative corrections, Nucl. Phys. **B 767** (2007), 29.
- [10] R. P. Woodard, Avoiding dark energy with $1/r$ modifications of gravity, Lect. Notes Phys. **720** (2007), 403.
- [11] A. Smilga, Comments on the Dynamics of the Pais-Uhlenbeck Oscillator. Sigma **5** (2009), 017.
- [12] P. Bolokhov and M. Pospelov, Classification of dimension 5 Lorentz-violating interactions in the standard model, Phys.Rev **D 77** (2008), 025022.
- [13] V. Kosteletzky and M. Mewes, Electrodynamics with Lorentz- Violating Operators of Arbitrary dimension, Phys.Rev. **D 80** (2009), 015020.
- [14] E. A. Bergshoeff, S. de Haan, W. Merbis, M. Porrati and J. Rosseel, Unitary Truncations and Critical Gravity: a Toy Model, JHEP **1204**, (2012), 134.
- [15] A. Dector, H. Morales- T  cotl, L. F. Urrutia and J. D. Vergara, An Alternative Canonical Approach to the Ghost Problem in a complexified Extension of the Pais-Uhlenbeck Oscillator, SIGMA **5** (2009), 053.
- [16] Margalli, C. A. and Vergara, J. D., Quantization of the Interacting Non-Hermitian Higher Order Derivative Field, arXiv:1309.2928 [hep-th].
- [17] F. A. Schaposnik, Chiral Symmetry in The Path Integral Approach, CNPq/CBPF (1986), 15-21.
- [18] P. A. M. Dirac, Lectures on Quantum Mechanics, Dover, New York, 2001.
- [19] M. Henneaux and C. Teitelboim, Quantization of Gauge Systems, Princeton University Press, Princeton, 1992.
- [20] M. V. Ostrogradski, M  moires sur les   quations differentielles relatives au probl  me des isop  rim  tres. Mem. Acad. St. Petersburg **6** (1850), 385.
- [21] W. Dittrich and M. Reuter, Classical and Quantum Dynamics: From Classical Paths to Path Integrals, Springer, Berlin, 2001.
- [22] A. Ashtekar, Lectures on Non Perturbative Canonical Gravity. World Scientific, Singapore, 1991.

- [23] H. A. Morales-Tecotl, L. F. Urrutia and J. D. Vergara, Reality conditions for Ashtekar variables as Dirac constraints, *Class. Quant. Grav.* **13** (1996), 2933.
- [24] V. O. Rivelles, Triviality of higher derivative theories, *Phys. Lett. B* **577** (2003), 137.
- [25] M. Faizal and M. Khan, A superspace formulation of the BV action for higher derivative theories, *Eur. Phys. J. C* **71** (2011), 1603.
- [26] K. Andrzejewski, J. Gonera, P. Machalski and K. Bolonek-Lasoń, On the triviality of higher-derivative theories, *Phys. Lett. B* **706** (2012), 427.
- [27] P. Senjanovic, Path Integral Quantization of Field Theories with Second-Class Constraints, *Annals of Physics*, 100, 227-261 (1976).
- [28] J.H. Lowenstein and J. A. Swieca, *Annals of Physics*, **68** (1971), 172.

INSTITUTO DE CIENCIAS NUCLEARES, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, APARTADO
POSTAL 70-543, MÉXICO 04510 DF, MÉXICO.

E-mail address: `carlos.margalli@nucleares.unam.mx`

E-mail address: `vergara@nucleares.unam.mx`